Which generic torus orbits in flag varieties are $\mathbb{Q}$–Gorenstein Fano?

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19 mai 2022

Toric varieties

- A toric variety X is a normal algebraic variety with an action of a torus $T \simeq (\mathbb{C}^*)^n$ such that X contains an open orbit isomorphic to T.
- Let $\Lambda \simeq \mathbb{Z}^n$ be a lattice; a fan Σ is a finite set of strictly convex polyhedral cones in the space $\Lambda_\mathbb{R} = \Lambda \otimes_\mathbb{Z} \mathbb{R}$ such that:
  (i) If $\sigma \in \Sigma$ and if τ is a face of σ, then $\tau \in \Sigma$;
  (ii) If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a common face of σ and τ.
- There is a one to one correspondence between fans and toric varieties; if Σ is a fan, we denote X_Σ the associated toric variety. In this talk we considered uniquely complete toric variety (*i.e.* the fan cover the whole space $\Lambda_\mathbb{R}$)

Roots system

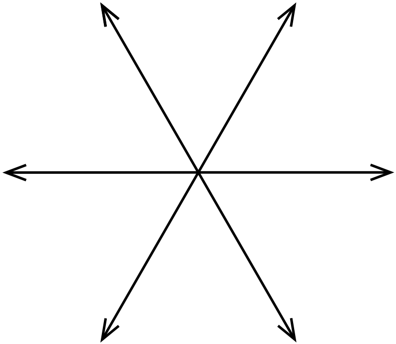
Let V be an euclidean space, a roots system R is a finite set of non zero vectors which generate V and such that:

(i) if $c \in \mathbb{R}$, $\alpha \in R$ and $c\alpha \in R$, then $c \pm 1$;

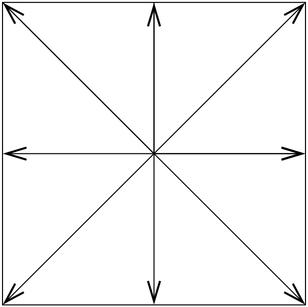
(ii) for all α, let s_α be the orthogonal reflection in the hyperplane $\alpha^\perp$, then $s_\alpha(R) = R$;

(iii) for all $\alpha, \beta \in R$, $s_\alpha(\beta) - \beta$ is an integer multiple of α.

Some roots system in dimension 2



(a) A_2

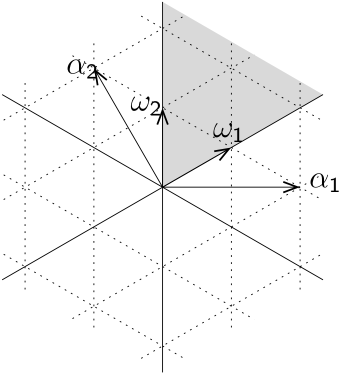


(b) B_2

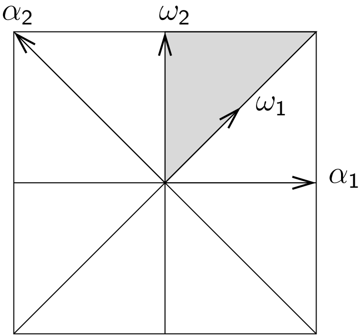
Objects associated to roots system R

- the Weyl group which is the group generated by the set $\{s_\alpha : \alpha \in R\}$;
- the dual roots system $R^\vee = \left\{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} : \alpha \in R\right\}$.
- the lattice generated by the roots: Λ_R (the roots lattice);
- the weights lattice $\Lambda_P = \{v \in V : \forall\alpha \ \ (v, \alpha^\vee) \in \mathbb{Z}\} \supset \Lambda_R$;
- Weyl chambers which are the closure of connected components of $V \setminus \bigcup\limits_{\alpha\in R} \alpha^\perp$;
- to a choice of a Weyl chamber $\mathcal{D}$ (the dominant chamber), we can define:
  (i) The set of fundamental weights: $\{\omega_1, \ldots, \omega_n\}$ ($\mathcal{D} = \sum_i \mathbb{R}^+ \omega_i$, and $(\omega_i)_i$ is a basis of the weights lattice); the weights in $\mathcal{D}$ are called dominant weights;
  (ii) The set of simple roots: $S = \{\alpha_1, \ldots, \alpha_n\}$ (basis of the roots lattice).

Some roots system in dimension 2



(c) A_2



(d) B_2

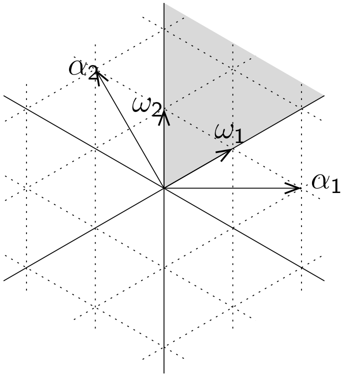
Fan associated to a subset L of the set of simple roots S. (construction of Klyachko and Vokrensenskii '84)

- We suppose that R is irreducible (and of dimension n).
- To each subset $L \subset S$ we can define $W_L =< s_\alpha : \alpha \in L >$ (so $W_S = W$);
- $\sigma_{R,L} = \bigcup_{w \in W_L} w\mathcal{D} \subset (\Lambda_P)_\mathbb{R}$
- Except in the trivial case $L = S$ the cone $\sigma_{R,L}$ is strictly convex. We suppose now that $L \neq S$.
- $\Sigma_{R,L}$ is the fan such that the cones of maximal dimension are:

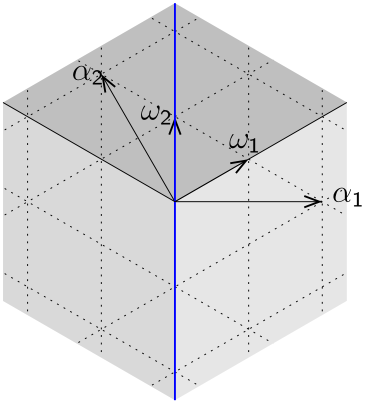
$$\Sigma_{R,L}(n) = \{w\sigma_{R,L} : w \in W\}.$$

- Let $X_{R,L}$ be the the toric variety associated to the fan $\Sigma_{R,L}$.

Exemples in dimension 2



(e) $R = A_2, L = \emptyset$



(f) $R = A_2, L = \{\alpha_1\}$

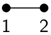
The link with generic torus orbit in flag varieties

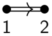
- To R corresponds a simple algebraic group G ; to L corresponds a parabolic subgroup $P \subset G$ (G/P is projective); the maximal torus T of G acts on G/P.

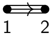
Theorem (Dabrowski '96)

If an orbit $T.x$ is generic then $\overline{T.x}$ is a normal variety and the fan of the toric variety $\overline{T.x}$ is equal to $\Sigma_{R^\vee,L}$.

List of varieties $X_{R,L}$ which are $\mathbb{Q}$-Gorenstein Fano

- An algebraic variety X is $\mathbb{Q}$-Gorenstein Fano if the canonical divisor K_X is $\mathbb{Q}$-Cartier and if $-K_X$ is ample.
- If X is $\mathbb{Q}$-Gorenstein Fano, the Gorenstein index of X is : $\min\{j > 0,\ jK_X \text{ is Cartier}\}$.
- $\mathbb{Q}$-Gorenstein Fano of index 1: Gorenstein Fano
- Gorenstein Fano$+$ smooth: Fano
- The irreducible roots systems are classified by Dynkin diagrams: a graph whose vertices are elements of S and edges depend on the angle between the two roots.
- • • $\pi/2$ 

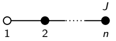
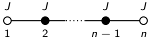
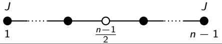
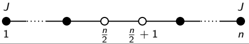
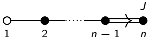
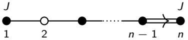
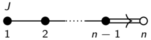
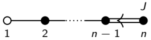
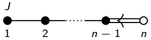
$2\pi/3$
- 

$3\pi/4$ 

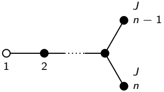
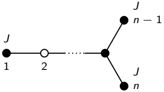
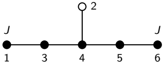
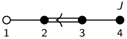
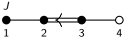
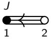
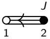
$5\pi/6$

List of varieties $X_{R,L}$ which are $\mathbb{Q}$–Gorenstein Fano (Rittatore-M.).

$\bullet$: elements in L

type(R)	rank	Dynkin(R), L and J	Geometry	$\varphi_{n_L} \in (\Lambda_{R^\vee})_{\mathbb{Q}}$
A_n	$n \geq 1$		Fano	$(n+1)\omega_1$
	$n \geq 2$		GF	$\omega_1 + \omega_n$
	n odd, $n \geq 3$		GF	$2\omega_{\frac{n+1}{2}}$
	n even, $n \geq 4$		Fano	$(n+1)\left(\omega_{\frac{n}{2}} + \omega_{\frac{n}{2}+1}\right)$
B_n	$n \geq 2$		GF	$\omega_1^\vee$
			GF	$\omega_2^\vee$
			GF, n even $\mathbb{Q}$–GF, n odd	$\omega_n^\vee$
C_n	$n \geq 3$		GF	$\omega_1^\vee$
			Fano	$2\omega_n^\vee$

Which generic torus orbits in flag varieties are $\mathbb{Q}$–Gorenstein Fano?

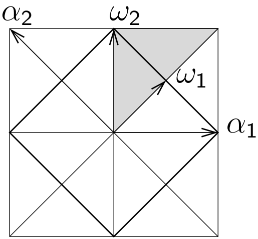
type(R)	rank	Dynkin(R), L and J	Geometry	$\varphi_{n_L} \in (\Lambda_{R^\vee})_{\mathbb{Q}}$
D_n	$n \geq 4$		Gorenstein Fano	$2\omega_1$
			Gorenstein Fano	ω_2
E_6	6		Gorenstein Fano	ω_2
F_4	4		$\mathbb{Q}$–Gorenstein Fano	$\frac{1}{2}\omega_1^\vee$
			Gorenstein Fano	$\omega_4^\vee$
G_2	2		Fano	$\omega_2^\vee$
			$\mathbb{Q}$–Gorenstein Fano	$\frac{1}{3}\omega_1^\vee$

This list includes the Fano cases which have been classified by Klyachko and Vokrensenskii ('84).

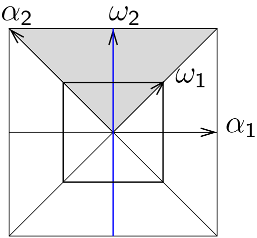
The beginning of the proof

- Let X_Σ be a toric variety of dimension n, then we define:
- $\Sigma(d) := \{\sigma \in \Sigma : \sigma \text{ of dimension } d\}$
- To each cone $\rho \in \Sigma(1)$:
  (i) D_ρ: the T-stable codimension one variety;
  (ii) u_ρ: the generator of $\rho \cap \Lambda$.
- $-K_{X_\Sigma} = \sum_{\rho \in \Sigma(1)} D_\rho$.
- For each $\sigma \in \Sigma$ we define: $\mathrm{Prim}(\sigma) = \cup_{\rho \subset \sigma} u_\rho$.
- There is an equivalence between:
  (i) X_Σ is $\mathbb{Q}$–Gorenstein Fano;
  (ii) for every cone $\sigma \in \Sigma(n)$, there exists $\varphi_\sigma \in \Lambda^\vee_{\mathbb{Q}}$ such that $\langle \varphi_\sigma, v \rangle = -1$ for $v \in \mathrm{Prim}(\sigma)$ and $\langle \varphi_\sigma, w \rangle > -1$ for every $w \in \mathrm{Prim}(\Sigma) \setminus \mathrm{Prim}(\sigma)$.
- If X is $\mathbb{Q}$–Gorenstein Fano, its Gorenstein index is equal to $\min\{j > 0 : \forall \sigma \in \Sigma(n)\ j\varphi_\sigma(\Lambda) \in \mathbb{Z}\}$.

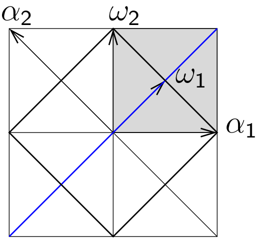
Example in for $R = B_2$



(g) $L = \emptyset$, Not ℚ–GF



(h) $L = \{\alpha_1\}$, Fano



(i) $L = \{\alpha_2\}$, GF

More details

- Let $n_{R,L}$ be a outward normal (for the scalar product) of the face $\mathrm{Conv}\,(\mathrm{Prim}(\sigma_{R,L}))$.

Proposition

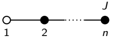
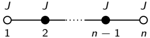
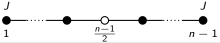
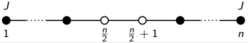
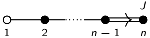
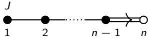
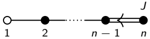
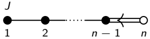
The variety $X_{R,L}$ is Gorenstein Fano if and only if $n_{R,L}$ belongs to the interior of $\sigma_{R,L}$.

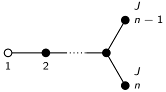
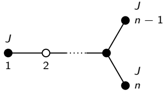
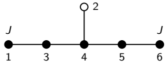
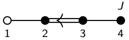
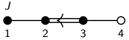
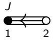
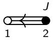
- To compute $n_{R,L}$ we have to describe the set of essential fundamental weights for $\sigma_{R,L}$ i.e. the fundamental weights which belongs to $\mathrm{Prim}(\sigma_{R,L})$.
- If λ is a dominant weight such that $\lambda = \sum_{i \in R \setminus L} a_i \omega_i$ with $a_i > 0$ then the normal fan of $\mathrm{Conv}\, W.\lambda$ is $\Sigma_{R,L}$
- The essential fundamental weights correspond to faces of codimension 1 of the Weyl polytope $\mathrm{Conv}\, W.\lambda$ which belongs to $\mathcal{D}$.
- These faces have been described in a work of Khare '17.

Associated Reflexive Polytopes

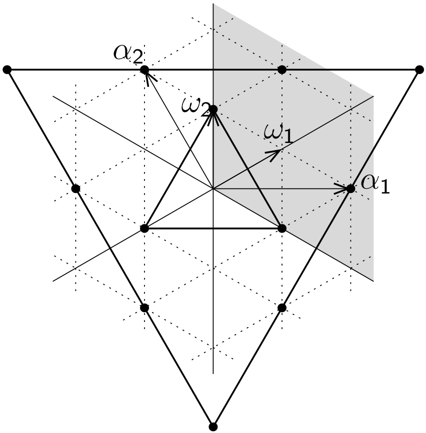
- Let $\mathcal{P} \in (\Lambda)_{\mathbb{R}}$ be a polytope containing 0 and with vertices in Λ; we define
$$\mathcal{P}^{\vee} = \{v \in (\Lambda^{\vee})_{\mathbb{R}} : \forall u \in \mathcal{P} \ \langle u, v\rangle \geq -1\}.$$
- If all vertices of $\mathcal{P}^{\vee}$ belong to $\Lambda^{\vee}$, $\mathcal{P}$ is called reflexive.
- To each Gorenstein Fano variety corresponds a pair of reflexive polytopes.
- We compute pairs corresponding to varieties $X_{R,L}$ which are Gorenstein Fano.
- On the one hand we have the polytope: $\mathrm{Conv}\,(W\{\omega_i : \omega_i \text{ essential }\}) \subset (\Lambda_P)_{\mathbb{R}}$
- On the other hand the Weyl polytope: $\mathrm{Conv}\,(W\varphi_{n_L}) \subset (\Lambda_{R^\vee})_{\mathbb{R}}$, where $\varphi_{n_L} \in (\Lambda_P)^{\vee} = \Lambda_{R^\vee}$ is the normal $\varphi_{n_L} \in (\Lambda_{R^\vee})_{\mathbb{R}}$ of $\mathrm{Conv}\,(\mathrm{Prim}(\sigma))$ such that $\varphi_{n_L}(\omega) = 1$ for ω essential.

J: fundamental weights which are essential $\bullet$: elements in L

type(R)	rank	Dynkin(R), L and J	Geometry	$\varphi_{n_L} \in (\Lambda_{R^\vee})_{\mathbb{Q}}$
A_n	$n \geq 1$		Smooth	$(n+1)\omega_1$
	$n \geq 2$		GF	$\omega_1 + \omega_n$
	n odd, $n \geq 3$		GF	$2\omega_{\frac{n+1}{2}}$
	n even, $n \geq 4$		Smooth	$(n+1)\left(\omega_{\frac{n}{2}} + \omega_{\frac{n}{2}+1}\right)$
B_n	$n \geq 2$		GF	$\omega_1^\vee$
			GF	$\omega_2^\vee$
			GF, n even $\mathbb{Q}$–GF, n odd	$\omega_n^\vee$
C_n	$n \geq 3$		GF	$\omega_1^\vee$
			Smooth	$2\omega_n^\vee$

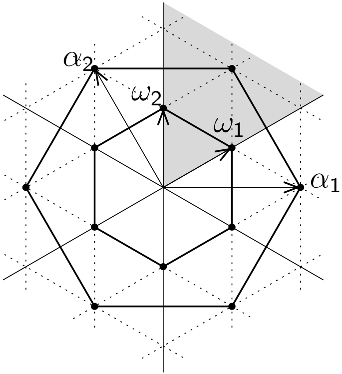
type(G)	rank	Dynkin(R), L and J	Geometry	$\varphi_{n_L} \in (\Lambda_{R^\vee})_{\mathbb{Q}}$
D_n	$n \geq 4$		Gorenstein Fano	$2\omega_1$
			Gorenstein Fano	ω_2
E_6	6		Gorenstein Fano	ω_2
F_4	4		$\mathbb{Q}$–Gorenstein Fano	$\frac{1}{2}\omega_1^\vee$
			Gorenstein Fano	$\omega_4^\vee$
G_2	2		Smooth, Fano	$\omega_2^\vee$
			$\mathbb{Q}$–Gorenstein Fano	$\frac{1}{3}\omega_1^\vee$

Exemples of reflexive polytopes in dimension 2



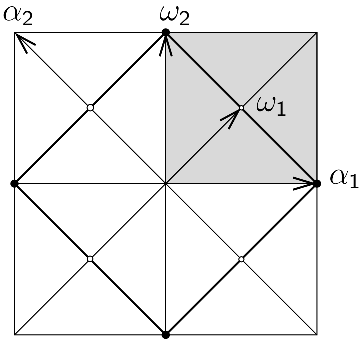
(j) $R = A_2, L = \{1\}$

Exemples of reflexive polytopes in dimension 2



(k) $R = A_2, L = \emptyset$

Examples of reflexive polytopes in dimension 2



(I) $R = B_2, L = \{2\}$

Thank You!